Int. J. Solids Structures, 1971, Vol. 7, pp. 5 to 10. Pergamon Press. Printed in Great Britain

THE GROWTH OF ONE-DIMENSIONAL SHOCK WAVES IN ELASTIC NONCONDUCTORS

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INTRODUCTION

IN THIS paper we study the propagation of shock waves in elastic materials that do not conduct heat. We derive a differential equation relating the strain and strain gradient behind the wave when the region ahead is unstrained and at constant entropy. Using this equation we are able to give conditions under which the wave will grow or decay at a given time. Generally, the results are qualitatively the same as in the purely mechanical theory. However, we show that for situations in which (i) the tangent modulus increases with temperature, (ii) the shock is strong, and (iii) the temperature behind the shock is low, the results are exactly opposite to those predicted by the mechanical theory.

1. CONSTITUTIVE ASSUMPTIONS

We consider the motion of a one-dimensional homogeneous elastic nonconductor defined by the constitutive relations

$$\begin{aligned} e &= \hat{e}(\varepsilon, s), \\ \sigma &= \hat{\sigma}(\varepsilon, s), \\ \theta &= \hat{\theta}(\varepsilon, s), \end{aligned} \tag{1.1}$$

where e is the internal energy, σ the stress, θ the absolute temperature, s the entropy and ε the strain. Here

$$\varepsilon = u_X, \tag{1.2}$$†

where $u = u(X, t)$ is the displacement of the material point X at time t. For convenience, we label material points by the positions they occupy in a fixed homogeneous reference configuration and suppose that the quantities e and s are measured per unit volume in this configuration.

† Subscripts denote partial differentiation with respect to the corresponding variable.

We assume that the response functions $\hat{e}$, $\hat{\sigma}$ and $\hat{\theta}$ are of class C^2. Then, as is well known, the second law requires that

$$\begin{aligned} \hat{\sigma} &= \hat{e}_{\varepsilon}, \\ \hat{\theta} &= \hat{e}_{s}. \end{aligned} \tag{1.3}$$

We call the quantities

$$E = \hat{\sigma}_{\varepsilon}, \qquad G = \hat{\sigma}_{s} \tag{1.4}$$

the *tangent modulus* and the *stress–entropy modulus*, respectively, and assume that

$$E > 0, \qquad G \neq 0. \tag{1.5}$$

Of course, E and G are functions of strain and entropy.

2. GENERAL PROPERTIES OF SHOCK WAVES

We assume that the motion contains a shock wave moving with (intrinsic) velocity

$$U = U(t) = \frac{\mathrm{d}Y(t)}{\mathrm{d}t},$$

where $Y(t)$ is the material point at which the wave is located at time t. Let f denote either ε, $\dot{u}$, or s. It then follows that $f, \dot{f}$ and f_X suffer jump discontinuities across the wave, but are continuous everywhere else. In view of the constitutive assumption (1.1) and the assumed smoothness of the response functions, the above assertion also holds for f equal to e, σ, or θ. In addition, we have the well known compatibility relations

$$U[\varepsilon] = -[\dot{u}], \tag{2.1}$$

$$\frac{\mathrm{d}[f]}{\mathrm{d}t} = [\dot{f}] + U[f_X]; \tag{2.2}$$

here we have used the standard notation for the jump $[f]$ in a function $f(X, t)$; i.e.

$$[f](t) = f^{-}(t) - f^{+}(t), \tag{2.3}$$

where

$$f^{\pm}(t) = \lim_{X \to Y(t)^{\pm}} f(X, t).$$

We assume that

$$U > 0; \tag{2.4}$$

thus f^{-} and f^{+} are the values of f immediately behind and just in front of the wave.

Let ρ denote the (constant) density in the reference configuration. Then the balance laws for momentum and energy imply that

$$\begin{aligned} [\sigma] &= -\rho U[\dot{u}], \\ [\sigma_X] &= \rho[\ddot{u}]. \\ -U[e + \tfrac{1}{2}\rho\dot{u}^2] &= [\sigma\dot{u}], \\ [\dot{e}] &= [\sigma\dot{\varepsilon}]. \end{aligned} \tag{2.5}$$

In deriving (2.5) we have assumed that there is no heat conduction, and that the external heat supply and body force vanish.

Equations (2.1) and $(2.5)_1$ imply the well known result

$$\rho U^2 = \frac{[\sigma]}{[\varepsilon]} \tag{2.6}$$

for the velocity of the shock, while (2.1), (2.2) with $f = \varepsilon$ and $f = \dot{u}$, and $(2.5)_2$ yield the relation

$$2U\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + [\varepsilon]\frac{\mathrm{d}U}{\mathrm{d}t} = U^2[\varepsilon_X] - \frac{1}{\rho}[\sigma_X]. \tag{2.7}$$

In view of $(1.1)_2$, (1.4) and the chain-rule, we can write (2.7) in the alternative form

$$2U\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + [\varepsilon]\frac{\mathrm{d}U}{\mathrm{d}t} = U^2[\varepsilon_X] - \frac{1}{\rho}\{[E\varepsilon_X] + [Gs_X]\}. \tag{2.8}$$

In addition, $(1.1)_1$, (1.3) and $(2.5)_4$ yield the familiar relation

$$[\theta\dot{s}] = 0. \tag{2.9}$$

3. A SHOCK WAVE ENTERING MATERIAL IN A HOMOGENEOUS STATE

We now assume that the material ahead of the wave is in a state of zero strain and constant entropy. Then

$$\varepsilon^+ = \dot{u}^+ = \varepsilon_X^+ = \dot{\varepsilon}^+ = \dot{s}^+ = s_X^+ = 0, \tag{3.1}$$

so that

$$[f] = f^- \tag{3.2}$$

when $f = \varepsilon$, $\dot{u}$, ε_x, $\dot{\varepsilon}$, $\dot{s}$, or s_X. By (3.2), equations (2.8) and (2.9) take the forms

$$\begin{gathered} 2U\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + [\varepsilon]\frac{\mathrm{d}U}{\mathrm{d}t} = \left(U^2 - \frac{E^-}{\rho}\right)[\varepsilon_X] - \frac{G^-}{\rho}[s_X], \\ [\dot{s}] = 0; \end{gathered} \tag{3.3}$$

hence (2.2) with $f = s$ implies

$$\frac{\mathrm{d}[s]}{\mathrm{d}t} = U[s_X]. \tag{3.4}$$

Our ultimate goal is to derive a relation between $\mathrm{d}[\varepsilon]/\mathrm{d}t$ and $[\varepsilon_X]$; we will accomplish this by expressing the terms in $(3.3)_1$ involving $\mathrm{d}U/\mathrm{d}t$ and $[s_X]$ as functions of $\mathrm{d}[\varepsilon]/\mathrm{d}t$. Our first step will be to derive a relation for $[s_X]$. By (3.2),

$$[\sigma\dot{\varepsilon}] = \sigma^-\dot{\varepsilon}^- = \sigma^-[\dot{\varepsilon}]$$

analogous assertions apply to the other product terms in (2.5). Therefore, using (2.1) and (2.2) with $f = \varepsilon$, we can write $(2.5)_{3,4}$ in the forms

$$\begin{aligned} [e] + \tfrac{1}{2}\rho U^2 [\varepsilon]^2 &= \sigma^- [\varepsilon], \\ [\dot{e}] &= \sigma^- \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} - \sigma^- U [\varepsilon_X]. \end{aligned} \tag{3.5}$$

By $(1.1)_1$, (1.3) and the chain-rule,

$$[e_X] = \sigma^- [\varepsilon_X] + \theta^- [s_X], \tag{3.6}$$

and (2.2) with $f = e$, $(3.5)_2$ and (3.6) imply that

$$\frac{\mathrm{d}[e]}{\mathrm{d}t} = \sigma^- \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + \theta^- U [s_X]. \tag{3.7}$$

Next, if we differentiate $(3.5)_1$ with respect to t, we arrive at

$$\frac{\mathrm{d}[e]}{\mathrm{d}t} = (\sigma^- - \rho U^2 [\varepsilon]) \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + [\varepsilon] \frac{\mathrm{d}\sigma^-}{\mathrm{d}t} - \rho U [\varepsilon]^2 \frac{\mathrm{d}U}{\mathrm{d}t} \tag{3.8}$$

and (3.7), (3.8) yield

$$\theta^- U [s_X] = -\rho U^2 [\varepsilon] \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} - \rho U [\varepsilon]^2 \frac{\mathrm{d}U}{\mathrm{d}t} + [\varepsilon] \frac{\mathrm{d}\sigma^-}{\mathrm{d}t}. \tag{3.9}$$

By $(1.1)_2$, (1.4), (3.2) and (3.4),

$$\frac{\mathrm{d}\sigma^-}{\mathrm{d}t} = E^- \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + G^- U [s_X], \tag{3.10}$$

and this relation when substituted in (3.9) yields

$$U(\theta^- - G^- [\varepsilon]) [s_X] = -\rho U [\varepsilon]^2 \frac{\mathrm{d}U}{\mathrm{d}t} + (E^- - \rho U^2) [\varepsilon] \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}. \tag{3.11}$$

If we differentiate (2.6) with respect to time, we conclude, with the aid of (3.10), that

$$2\rho U \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{E^-(1-\mu)}{[\varepsilon]} \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + \frac{G^- U}{[\varepsilon]} [s_X], \tag{3.12}$$

where

$$\mu = \frac{\rho U^2}{E^-}. \tag{3.13}$$

Equations (3.11) and (3.12) imply, after some manipulation, that

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t} &= \frac{E^-(1-\mu)\tau}{\rho U (2\tau - 1)[\varepsilon]} \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}, \\ [s_X] &= \frac{E^-(1-\mu)}{G^- U (2\tau - 1)} \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}, \end{aligned} \tag{3.14}$$

where

$$\tau = \frac{\theta^-}{G^- [\varepsilon]}. \tag{3.15}$$

Finally, (3.3) and (3.14) yield the desired result†

$$\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} = -\frac{U(1-\mu)(2\tau-1)}{(3\mu+1)\tau-(3\mu-1)}[\varepsilon_X]. \tag{3.16}$$

4. COMPRESSIVE SHOCK WAVES

We now consider a compressive shock wave for which

$$[\varepsilon] = \varepsilon^- < 0. \tag{4.1}$$

We assume that the isentropic stress–strain law in compression is concave from below, i.e.

$$\hat{\sigma}_{\varepsilon\varepsilon}(\varepsilon, s) < 0 \tag{4.2}$$

for $\varepsilon \leq 0$ and all s. Then $(1.4)_1$, $(1.5)_1$, (2.6) and (4.2) imply that

$$0 < \mu < 1, \tag{4.3}$$

and a careful study of the right-hand side of (3.16) yields the following:

Theorem. Consider a compressive shock wave and assume that the material ahead of the wave is in a state of zero strain and constant entropy. Assume further that (4.2) *holds.*

(i) *If* $\tau > \frac{1}{2}$ *or* $\tau < (3\mu-1)/(3\mu+1)$, *then*

$$\begin{aligned} [\varepsilon_X] > 0 &\Leftrightarrow \frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} > 0, \\ [\varepsilon_X] < 0 &\Leftrightarrow \frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} < 0. \end{aligned} \tag{4.4}$$

(ii) *On the other hand, if* $(3\mu-1)/(3\mu+1) < \tau < \frac{1}{2}$, *then*

$$\begin{aligned} [\varepsilon_X] < 0 &\Leftrightarrow \frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} > 0, \\ [\varepsilon_X] > 0 &\Leftrightarrow \frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} < 0. \end{aligned} \tag{4.5}$$

We now show that in most situations τ will be greater than $\frac{1}{2}$ so that (4.4) will hold. Indeed, by (1.3) and (1.4),

$$G^- = \hat{\theta}_\varepsilon(\varepsilon^-, s^-), \tag{4.6}$$

and it follows from (3.15) that $\tau > \frac{1}{2}$ if and only if

$$\hat{\theta}(\varepsilon^-, s^-) - \tfrac{1}{2}\hat{\theta}_\varepsilon(\varepsilon^-, s^-)\varepsilon^- > 0. \tag{4.7}$$

† For an ideal gas a relation of this type was derived by Harris [1], for a nonlinear Maxwell material by Duvall and Alverson [5], and for a general nonlinear viscoelastic material by Chen and Gurtin [9]; all of the above papers neglect the influence of thermodynamics.

Since $\hat{\theta} > 0$ and $\varepsilon^- < 0$, if $G^- > 0$, then (4.7) will be satisfied. Thus assume $G^- < 0$, which is the case for most materials. Then (4.7) will remain valid provided

$$\hat{\theta}_{\varepsilon\varepsilon}(\varepsilon, s) < 0, \tag{4.8}$$

or equivalently, by (1.3) and $(1.4)_1$,

$$E_s(\varepsilon, s) < 0. \tag{4.9}$$

Therefore (4.4) holds for most metals, since the tangent modulus is a decreasing function of entropy for such materials.† On the other hand, for a gas the tangent modulus usually increases with temperature. For such materials, even though (4.9) is violated, (4.7) and hence (4.4) will remain valid provided the temperature $\hat{\theta}(\varepsilon^-, s^-)$ is sufficiently high and the strain $|\varepsilon^-|$ sufficiently small. Thus we would expect (4.4) to be satisfied except in cases for which all of the following hold: (i) $E_s(\varepsilon, s) > 0$; (ii) the shock is strong; and (iii) the temperature behind the shock is low.

Acknowledgements—We wish to thank Darrell E. Munson for valuable discussions. This research was supported in part by the U.S. Atomic Energy Commission and in part by the U.S. National Science Foundation.

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(*Received* 4 *March* 1970)

† See, for example, Sutton [2] and Kamm and Alers [6] (aluminum), Lord and Beshers [8] (iron), Overton and Gaffney [3] (copper), Smith and Arbogast [4] (beryllium), Lowrie and Gonas [7] (tungsten).